Dynamical "breaking" of time reversal symmetry and converse quantum ergodicity

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Abstract

It is a common assumption that quantum systems with time reversal invariance and classically chaotic dynamics have energy spectra distributed according to GOE-type of statistics. Here we present a class of systems which fail to follow this rule. We show that for convex billiards of constant width with time reversal symmetry and "almost" chaotic dynamics the energy level distribution is of GUE-type. The effect is due to the lack of ergodicity in the "momentum" part of the phase space and, as we argue, is generic in two dimensions. Besides, we show that certain billiards of constant width in multiply connected domains are of interest in relation to the quantum ergodicity problem. These billiards are quantum ergodic, but not classically ergodic.

1 Introduction

The famous conjecture of Bohigas, Giannoni and Schmit (BGS) [1] asserts that the energy levels of classically chaotic systems are distributed as eigenvalues of random-matrix ensembles. Accordingly, the statistics of the energy levels is universal and depend only on symmetries of the system. That means the energy levels distribution for (spinless) chaotic systems with time-reversal invariance should be close to that of Gaussian Orthogonal Ensemble (GOE). If the time-reversal invariance is broken the distribution of the energy levels follows statistics of Gaussian Unitary Ensemble (GUE). The BGS conjecture have been supported by broad numerical and experimental evidence. Indeed, for a large number of systems without additional symmetries spectral statistics are in agreement with the above predictions. This might not be true, however, if additional symmetries are present. Examples of symmetric billiards with anomalous spectral statistics were given in [2], [3]. These billiards are time-reversal invariant, but in addition have a certain rotational symmetry. As a result, the statistics for a part of their spectra turns out to be of GUE-type (rather than of GOE-type). On the other hand, anomalous spectral statistics may also appear in systems with broken time reversal invariance. For example, the energy levels of magnetic billiards with reflection symmetry are known to be distributed as in GOE [4].

In this paper we introduce a class of convex billiards of constant width with smooth boundaries whose dynamics are time-reversal invariant and "almost" chaotic. These billiards have no additional symmetries but, nevertheless, exhibit statistics of GUE-type. On the contrary, if an additional reflectional symmetry is present, the spectral statistics turns to be of GOE-type. We then give an elementary semiclassical explanation for these results and discuss the implications for spectra of generic convex billiards with smooth boundaries.

In the second part of the paper we deal with fully chaotic billiards of constant width in the multiply connected domains. As we explain below, the spectral statistics of these billiards differ from those of smooth convex billiards of constant width. Nevertheless, it has been recently shown in [5] that for large energy ranges the spectrum exhibits GUE like behavior, as well. Our interest to these billiards here, is primely related to the converse quantum ergodicity problem. By Schnirelman theorem [6] classical ergodicity implies quantum ergodicity. The converse question make sense too [7]: Are quantum ergodic systems necessarily classical ergodic? As we demonstrate, certain symmetric billiards of the above type provide negative answer to this question. In other words, they are quantum ergodic but not classically. To the best of our knowlage this is the first example of such systems.

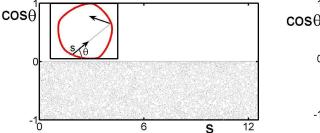
2 Billiards of constant width

We deal with a class of convex billiard tables of constant width with smooth boundaries. That means the maximal distance between any point on the billiard boundary and other boundary points is a constant. There is a simple way to construct such billiards by using the following parameterization for the billiard boundaries $z(\alpha)$ in the complex plane:

$$z(\alpha) = z(0) - i \sum_{n \in \mathbb{Z}} \frac{a_n}{n+1} \left(e^{i\alpha(n+1)} - 1 \right), \qquad \alpha \in [0, 2\pi).$$
 (1)

Here $z(\alpha)$ defines a curve of constant width, whenever the parameters a_n 's satisfy conditions: $a_{-n} = a_n^*$, $a_1 = 0$, $a_{2n} = 0$ for n > 0 [8]. Previously, billiards of constant width have attracted an attention due to their unusual geometry of caustics. Our interest here stems from their peculiar dynamical property: a billiard trajectory which hits the boundary with an angle in the interval $[0, \pi/2]$ (resp. $[\pi/2, \pi]$) must hit the boundary at the next bouncing point with an angle at the same interval. In other words, the billiard ball once launched clockwise (resp. anti-clockwise) will move in that way forever.

The billiard dynamics can be described in a standard way with the help of the associated Poincare map. The map acts on unit vectors attached to the boundary by translating them according to the rules of billiard dynamics. The corresponding two-dimensional phase space can be parameterized by a couple of coordinates prescribing position and direction of the unite vectors. The canonical choice is $(s, \cos \theta)$, where $s \in [0, 2\pi a_0)$ is the arclength parameter along the boundary and $\theta \in [0, \pi]$ is the angle between the unite vector and the tangent line to the boundary. In the case of time reversal invariant dynamics the phase space has a reflectional symmetry along the line $\theta = \pi/2$. Furthermore, for billiards of constant width this symmetry line is invariant under the billiard map and separates the motions in the clockwise and anti-clockwise directions, see fig. 1. In general, a billiard of constant width defined by eq. 1 has a mixed phase space where regions of regular motion, i.e., Kolmogorov-Arnold-Moser (KAM) tori and elliptic islands coexist with regions of chaotic motion. In particular, in the vicinity of the line $\theta = \pi/2$ there



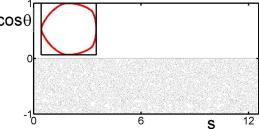


Figure 1: The insets show non-symmetric (left) and symmetric (right) billiards of constant width with parameters $\{a_0 = 2, a_3 = i/4, a_5 = 1/2 + i/2, a_{2k+1} = 0, k > 2\}$ and $\{a_0 = 2, a_3 = i/4, a_5 = -3i/4, a_{2k+1} = 0, k > 2\}$ respectively. The corresponding phase space pictures are obtained after hundreds of iterations of the billiard map applied to a number of initial points located in the lower $(\theta > \pi/2)$ half of the phase space. Note, that the iterated points do not penetrate into the upper half. This implies separation of the clockwise and anti-clockwise types of motion. (For a comparison with a generic billiard see fig. 4)

always exists a region filled by KAM tori. For our purposes, it will make sense to consider billiards with "maximally" chaotic phase space. Two such billiards are shown in fig. 1. Here the parameters a_n are adjusted in a way to minimize the sizes of elliptic islands, as well as, regions of KAM tori along the lines $\theta = \frac{\pi}{2}$ and $\theta = 0, \pi$ (whispering gallery region).

3 Spectral statistics

We consider the energy spectrum of the quantum billiards in fig. 1 with the Dirichlet boundary conditions. As has been explained above, the change from clockwise to anticlockwise types of motion is classically forbidden process. On the other hand, in the quantum billiards such a switch is possible due to the tunneling effect. The corresponding tunneling time τ is finite and depends on the width of the "dynamical barrier" along the separation line $\theta = \frac{\pi}{2}$. This leads to a special structure of the energy spectrum. Most of the energy levels can be separated into quasidegenerate pairs $\{E_n^s, E_n^a\}$. Here E_n^s , E_n^a correspond to the symmetric $\varphi_n^s \approx (\varphi_n^l + \varphi_n^r)/\sqrt{2}$ and the antisymmetric $\varphi_n^a \approx (\varphi_n^l - \varphi_n^r)/\sqrt{2}$ combinations of the clockwise φ_n^r and the anti-clockwise φ_n^l quasimodes. The Wigner transforms of φ_n^r , φ_n^l are entirely concentrated in the "lower" $(\theta < \pi/2)$ and the "upper" $(\theta > \pi/2)$ half of the phase space, respectively. Furthermore, the splittings of the energy levels $\delta E_n = |E_n^a - E_n^s|$ determine the time $\tau \sim \hbar/\langle \delta E_n \rangle$ needed for a wavepacket to switch the direction of motion. Since the tunneling time through the "dynamical barrier" is exponentially large in \hbar , this results in exponentially small splittings δE_n between the energy levels. The rest of the spectrum $\{E_n^0\}$ contains eigenfunctions φ_n^0 whose Wigner transform localized in the invariant neighbourghood of the line $\theta = \pi/2$. For the considered range of energies the only visible part of $\{E_n^0\}$ are unpaired zero angular momentum bouncing modes localized in the phase space exactly on the separation line.

Since all paired levels (E_n^s, E_n^a) are quasi-degenerate it makes sense to consider half of the spectra, e.g., E_n^s . (The levels E_n^0 constitutes a tiny fraction of the whole spectrum and

have no significant impact on the spectral statistics anyway.) Our primary interest is the spectral statistics of the energy levels E_n^s . To this end we have numerically calculated by the scaling method of Vergini and Saraceno [9] a number (~ 15000) of the energy levels for each of the billiards in fig. 1. The results for the nearest-neighbouring distribution P(s) are peresented in fig. 2. As one can clearly see, the distribution for the billiard without additional symmetries (left in fig. 1) clearly follows the pattern of GUE. This contradicts a common belief that chaotic systems with time reversal invariance have spectra of GOE type when additional symmetries are absent. In contrast, the distribution P(E) for the billiard with a reflectional symmetry (right in fig. 1) exhibits GOE type of statistics. Below we provide an elementary explanation for these results based on the semiclassical link between spectral statistics and periodic orbits of the system. Specifically, let us focus on the spectral form factor K(T). It is defined as the Fourier transform of the autocorrelation function

$$R(s) = \bar{d}^{-2} \langle d(E+s)d(s) \rangle - 1, \tag{2}$$

where $d(E) = \sum \delta(E - E_n)$, $\bar{d} = \langle d(E) \rangle$ denote the density of states and its mean value. By means of the semiclassical trace formula the density of states can be written as a sum $d(E) = \bar{d} + \sum A_n \exp(iS_n(E)/\hbar)$ over periodic orbits, where phases $S_n(E)$ include both actions and Maslov indices of the periodic orbits. After substitution of d(E) into (2) and taking the Fourier transform one gets the semiclassical representation of K(T) as double sum over pairs of periodic orbits. The spectral form factor can be naturally separated $K(T) = K_{diag}(T) + K_{off}(T)$ into two terms provided by diagonal $(S_i = S_j)$ and off-diagonal $(S_i \neq S_j)$ correlations of periodic orbits. The leading diagonal term was

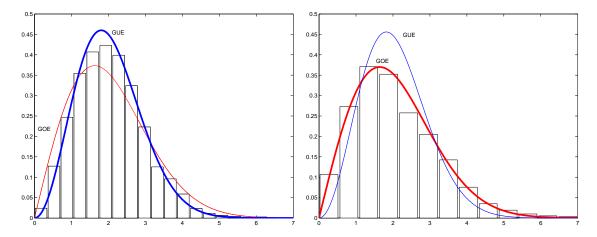


Figure 2: Nearest neighbor distribution of energy levels for nonsymmetric (left) and symmetric (right) billiards in fig. 1. For comparison, GOE and GUE predictions are shown as well.

derived by Berry [10] and in the Heisenberg time $T_H = 2\pi\hbar\bar{d}$ units $t = T/T_H$ found to be $K_{diag}(t) = \beta t$, with $\beta = 2$ for time reversal invariant systems and $\beta = 1$ otherwise. This should be compared with the spectral form factors

$$K_{GUE}(t) = t$$
, $K_{GOE}(t) = 2t + t \ln(2t + 1)$, for $t < 1$

for GUE and GOE. In the absence of time invariance K_{off} vanishes and the diagonal term alone reproduce correctly K_{GUE} . On the contrary, for time reversal invariant systems K_{diag} gives only leading term and the off-diagonal correlations between periodic orbits must provide the rest. It has been first shown in [11] that GOE result can indeed be reproduced correctly if one takes into consideration the correlations between pairs of selfencountered periodic trajectories which approach themself from the opposite directions under small angles. As a result, the term of order n > 1 in the Taylor expansion of $K_{GOE}(t)$ comes from the correlations of pairs of periodic orbits with n-1 self-encounters. It is a straightforward observation that pairs of self-encountered periodic orbits just do not exist in billiards of constant width, since trajectories cannot reverse their directions of motion, see fig. 3. This implies that $K_{off}(t)$ must be zero and $K(t) = K_{GUE}(t) \equiv t$. Hence, the spectral form factor of the non-symmetric billiard in fig. 1 (left) should be of GUE and not of GOE type. On the other hand, for the billiard in fig. 1 (right) the reflectional symmetry substitutes the role of time reversal invariance and restores correlations between periodic orbits. (The simplest way to see this is to consider a half of the billiard. The dynamics there are not "uni-directional" and self-encountered trajectories do exist.) That leads back to GOE type of spectral statistics. This is in complete analogy with the case of reflectional symmetric billiards in the presence of magnetic field, where one observes GOE statistics instead of GUE [4].

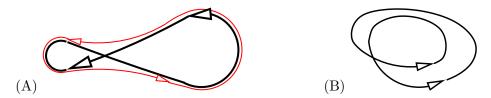


Figure 3: Sketch of a pair of self-encountered periodic orbits (A), and of "typical" periodic orbit in a billiard of constant width (B). Note that pairs of self-encountered periodic orbits do not appear in billiards of constant width.

4 Generic convex billiards

Let us consider now the implications of the above results for spectra of generic billiards whose dynamics is neither fully integrable nor chaotic. In that case, by the Berry-Robnik theory [12] the energy spectrum is composed of the independent spectra corresponding to the invariant parts of the phase space. Thus for systems with time reversal symmetry, one might expect the energy level distribution be a mixture of the Poissonian statistics with GOE type statistics related to regular and chaotic parts of dynamics, respectively. However, as we argue below, a general picture could be somewhat more complex. For a typical billiard with smooth boundaries the chaotic part of the phase space could be separated into two types of invariant regions, as shown in fig. 4. In the regions of the first type the dynamics are bi-directional. That means the billiard ball launched from such a region might reverse the direction of flight in the course of motion. As a result, the periodic trajectories admit self-encountering and one can expect the corresponding statistics be of GOE type. In the regions of the second type the dynamics are always

uni-directional. Here the full switch of flight direction is prevented by KAM tori and the resulting statistics should be of GUE type. Thus, in the absence of spatial symmetries the overall spectral statistics must be a mixture of independent GUE, GOE and Poissonian statistics corresponding to the invariant sets with uni-directional, bi-directional chaotic dynamics and regular dynamics. If an additional reflectional symmetry exists in the billiard then only GOE and Poissonian parts are present. This should describe a typical structure of spectra for two dimensional billiards with smooth boundaries. In more then two dimensions, however, KAM tori do not separate regions of phase space. So it could be expected that typically only bi-directional type of motion exists and only GOE type of subspectra appear.

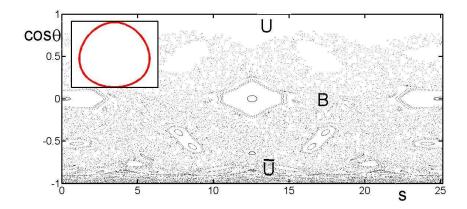


Figure 4: A "generic" billiard $(a_2 \neq 0)$ with parameters $a_0 = 4, a_2 = 0.1, a_3 = 0.5, a_5 = 0.1, a_k = 0, k > 5$ defined by eq. 1. The billiard map is applied to the initial points located in the lower half of the phase space. Note, that the iterated points do not penetrate into a certain domain U. Hence, U is dynamically separated from the symmetric domain \bar{U} and the corresponding dynamics are "uni-directional". On the contrary, the dynamics in the central part B of the phase space are "bi-directional".

It worth to notice that a mixture of large number of independent spectra would result in the Poissonian statistics, irrespectively of the statistics of individual components. Thus for generic systems it would be hard, in practice, to see the appearance of GUE type subspectra. The billiards of constant width represent a very special class of dynamical systems where bi-directional type of dynamics is completely absent and the effect can be clearly observed.

5 Fully chaotic billiards and converse quantum ergodicity

The billiards considered so far are only "approximately" chaotic. But it is also possible to construct fully chaotic billiards of constant width in multiply connected domains. An example of "hippodrome" like billiard is shown on fig. 5a. By Wojtkowski's cone field

method [14] it can be easily shown that the dynamics in this billiard are fully hyperbolic with a positive Lyapunov exponent almost everywhere. Note, however that the billiard is not ergodic, since there are exist (at least) two ergodic components corresponding to the clockwise and anti-clockwise types of motion separated by the invariant line. Despite of similar phase space structure, the quantum spectral properties of the billiards in fig. 1 and fig. 5 are essentially different. For fully chaotic billiards the width of separation region between two types of motion shrinks to zero and the tunneling time is determined by diffraction effects at the points of billiard boundary with curvature jumps. This results in much shorter (algebraic rather than exponential in \hbar) tunneling times. Thus instead of quasi-degeneracies, one can expect splitting between the energy levels be comparable with the mean level spacing. The spectral statistics of non-symmetric billiards of such type ("Monza billiards") have been recently investigated in [5]. It has been shown there, that the long range correlations among levels tend to exhibit GUE type behavior. This result is in agreement with our observations for convex billiards of constant width with smooth boundaries. In what follows, we will show that the "hippodrome" like billiards are also of interest in connection with the quantum ergodicity problem.

It is well known that classical ergodicity implies quantum. But is the converse also true [7, 13]: Are quantum ergodic systems necessarily classically ergodic? As we show below the "hippodrome" billiard provide a counterexample. This billiard is classically not ergodic. Let us show, however, that it is quantum ergodic. First, observe that by desymmetrization procedure the full spectrum of the billiard in fig. 5a can be decomposed into the spectra of four "quarter" billiards shown in fig. 5b. The "quarter" billiards have four different combinations $\mathcal{S} = \{DD, DN, ND, NN\}$ of Neumann and Dirichlet boundary conditions at two flat pieces of the boundary. We will use index $\nu \in \mathcal{S}$ to denote these billiards and call φ_n^{ν} the corresponding eigenfunctions. By definition, any eigenfunction φ_n of the full billiard is just collection of four copies of φ_i^{ν} for some ν and $n=\eta_{\nu}(i)$ with $\eta_{\nu}(\cdot): \mathbb{N} \to \mathbb{N}$ being the function which maps indices of ν 's quarter billiard eigenfunctions to the indices of the corresponding eigenfunctions of the full billiard. Note, that each of the desymetrized billiards is classically ergodic (assuming that there exist only two ergodic components in full billiard). Hence, by Schnirelman's theorem all four quantum billiards have eigenfunctions "equidistributed" in the phase space. More precisely, this means for each combination of boundary conditions ν there exists a subset of integers $\{j_n^{\nu}\}_{n\in\mathbb{N}}$ of counting density one such that the classical average of an observable A(x) coincides with the corresponding quantum limit:

$$\lim_{n \to \infty} \langle \varphi_{j_n^{\nu}}^{\nu}, \operatorname{Op}(A) \varphi_{j_n^{\nu}}^{\nu} \rangle = \int A(x) \, d\mu, \tag{3}$$

where μ stands for normalized Liouville measure in the billiard phase space and $\operatorname{Op}(A)$ means Weyl quantization of A. Now, let V be the phase space of the full billiard and let $V = \bigcup_{k=1}^4 V_k$ be its decomposition into four symmetric pieces. For an observable O in V let O_k denote its restriction to the quarter V_k , i.e., $O_k(x) = O(x)$ if $x \in V_k$. Then for the corresponding quantum averages one has decomposition

$$\langle \varphi_l, \operatorname{Op}(O) | \varphi_l \rangle = \sum_{k=1}^4 \langle \varphi_n^{\nu}, \operatorname{Op}(O_k) | \varphi_n^{\nu} \rangle, \qquad l = \eta_{\nu}(n),$$
 (4)

where O_k in the righthand side of eq. 4 should be understood as observables in the phase spaces of the quarter billiards. Now define the sequence of integers $\{j_n\} = \bigcup_{\nu} \{\eta_{\nu}(j_n^{\nu})\}$. This sequence is of density one and it follows immediately from eqs. 3,4:

$$\lim_{n \to \infty} \langle \varphi_{j_n}, \operatorname{Op}(O) \varphi_{j_n} \rangle = \sum_{k=1}^{4} \int_{V_k} O_k \, d\mu = \int_V O \, d\mu, \tag{5}$$

implying quantum ergodicity for classically non-ergodic billiard in fig. 5a.

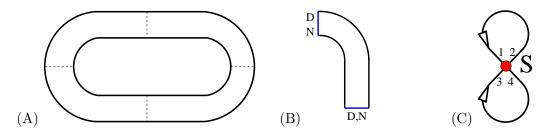


Figure 5: On the left: Fully chaotic "hippodrome" billiard (A) and its desymmetrized quarter (B) with different combinations of boundary conditions. Both the external and internal billiard walls are composed of the boundaries of stadia in a way that the billiard interior has a constant width. On the right (C): "Uni-directional" quantum graph. The scattering matrix at the vertex satisfies conditions: $S_{1,3} = S_{3,1} = S_{2,4} = S_{4,2} = 0$, $S_{i,i} = 0$, i = 1, ... 4

Let us briefly comment on the reasons for failure of classical ergodicity in the "hippodrome" like billiards. There is a general result stating two necessary and sufficient conditions for a billiard to be classically ergodic. Let $\{\lambda_n = \sqrt{E_n}/\hbar, \varphi_n\}$ be the billiard spectral data and $N(\lambda)$ be the corresponding counting function, then:

Theorem 1 [See [13] and references there.] The billiard flow is ergodic if and only if for every observable A:

Condition 1:
$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} |\langle \varphi_n, \operatorname{Op}(A) \varphi_n \rangle - \int A(x) d\mu|^2 = 0.$$

Condition 2:
$$\forall \epsilon, \exists \delta, \lim_{\lambda \to \infty} \sup \frac{1}{N(\lambda)} \sum_{\substack{n \neq k, \lambda_n, \lambda_k \leq \lambda \\ |\lambda_n - \lambda_k| < \delta}} |\langle \varphi_n, \operatorname{Op}(A) \varphi_k \rangle|^2 < \epsilon.$$

The first condition is essentially equivalent to the quantum ergodicity and as has been already shown is satisfied for the hippodrome billiard. Thus the second condition must fail. It is easy to understand the reason for that. Although there are no exponentially small degeneracies in the spectra of the billiard (as for convex billiards of constant width) its eigenfunctions could be still approximately separated into the pairs of symmetric and antisymmetric combinations:

$$\varphi_n^s = \frac{1}{\sqrt{2}}(\varphi_n^r + \varphi_n^l) + r_1, \qquad \varphi_n^a = \frac{1}{\sqrt{2}}(\varphi_n^r - \varphi_n^l) + r_2, \tag{6}$$

of clockwise φ_n^r and anti-clockwise φ_n^l moving quasimodes. Consider now the observable χ which is one for points of the phase space moving in the clockwise direction and zero

otherwise. If the reminder terms r_1, r_2 are sufficiently small (as has been numerically observed in [5]), the off-diagonal elements

$$\left|\left\langle \varphi_{n}^{s},\operatorname{Op}\left(\chi\right)\varphi_{n}^{a}\right\rangle \right|^{2}=\left|\frac{1}{2}+\frac{1}{\sqrt{2}}\left(\left\langle \varphi_{n}^{r},r_{2}\right\rangle +\left\langle r_{1},\varphi_{n}^{r}\right\rangle \right)+\left\langle r_{1},\operatorname{Op}\left(\chi\right)r_{2}\right\rangle \right|^{2}+O(\lambda^{-1})>C,$$

are bounded from below, and as a result Condition 2 is not satisfied.

6 Conclusions

There are two general conclusions which can be drawn from the present study of quantum billiards of constant width. The first one is that in the absence of additional symmetries, time reversal invariance of a chaotic system does not automatically guarantees GOE type statistics for the energy spectrum. In addition, the following dynamical condition must be satisfied. Call an invariant ergodic component D_i of the phase space "time reversally connected" if for (almost) every point $(q, p) \in D_i$ with the coordinate q and momentum p, the "reverse" point (q, -p) belongs to D_i too. Then (in the absence of additional symmetries) the energy spectrum associated with a chaotic invariant domain D is of GOE type only if D is "time reversally connected". Otherwise, the associated spectrum should be of GUE type. Loosely speaking, in the systems, such as billiards of constant width time reversal symmetry is broken dynamically, rather than with an external (e.g., magnetic) field. Beyond the spectral statistics, other quantum properties in these systems should be affected too. For instance, the semiclassical treatment of the Landauer conductance in [15] and short noise in [16] through ballistic devices rely on calculations of the correlations between selfencounted periodic trajectories. So one can use exactly the same arguments as above to derive GUE type results for quantum dots with non-symmetric shapes of constant width (resp. GOE for symmetric shapes).

The second conclusion is that classical ergodicity does not follow in general from quantum ergodicity alone. There exist chaotic systems like hippodrome billiards where the diagonal elements satisfy Condition 1 of the Theorem 1 (i.e., quantum ergodicity holds) but off-diagonal terms fail to satisfy Condition 2. This shows that in general some sort of additional condition on off-diagonal terms is, indeed necessary to grantee classical ergodicity.

Finally, it is worth mentioning that besides billiards, there exist other systems with unidirectional type of dynamics. For instance, the above billiard construction can be straightforwardly generalized to get a family of smooth "uni-directional" potentials. Namely, fix the coefficients $a_n, n > 0$ in eq. 1, set $z(0) = -ia_0$ and let a_0 vary over an interval $\Delta = [\delta_1, \delta_2], \ \delta_{1,2} > 0$. For an appropriate choice of δ_1, δ_2 (that means δ_1 cannot be too small) this defines a family of closed convex curves $\Gamma(a_0), a_0 \in \Delta$ of constant width in the domain bounded by $\Gamma(\delta_1)$ and $\Gamma(\delta_2)$. Now, let V(z) be a smooth potential which is equal ∞ outside $\Gamma(\delta_2)$, 0 (or ∞) inside $\Gamma(\delta_1)$ and whose equipotential lines coincide with $\Gamma(a_0), a_0 \in \Delta$ in between. Any such potential gives rise to the uni-directional Hamiltonian flow inside the domain bounded by $\Gamma(\delta_2)$. Another class of uni-directional systems is provided by quantum graphs of a certain type. A simple example is shown in fig. 5c. Here the full separation of two types of motion is achieved by putting appropriate scattering matrices at the vertices of the graph. Acknowledgments: I am grateful to J. Avron and S. Fishman for their support during my stay at Physics Department of Technion. I also would like to thank S. Nonnenmacher for helpful discussions on converse quantum ergodicity problem. This work was supported by Minerva Foundation.

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